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Ginsparg-Wilson fermions in odd dimensions

Wolfgang Bietenholz

Humboldt Universität, Institut für Physik
Invalidenstr. 110, D-10115 Berlin, Germany
E-mail: bietenho@physik.hu-berlin.de

Jun Nishimura*

The Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
E-mail: nisimura@nbi.dk

Abstract: The Ginsparg-Wilson relation, if written in a suitable form, can be used as a condition for lattice Dirac operators of massless fermions also in odd dimensions. The fermion action with such a Dirac operator is invariant under a generalized parity transformation, which reduces to the ordinary parity transformation in the (naive) continuum limit. The fermion measure, however, transforms non-trivially under the generalized parity transformation, and hence the parity anomaly arises solely from the fermion measure. The analogy to the lattice construction of chiral gauge theories in even dimensions is clarified by considering a dimensional reduction. We also propose a natural definition of a lattice Chern-Simons term, which is consistent with odd dimensional Ginsparg-Wilson fermions.

Keywords: Field Theories in Lower Dimensions, Lattice Gauge Field Theories, Space-Time Symmetries, Anomalies in Field and String Theories.

*On leave of absence from Department of Physics, Nagoya University, Nagoya 464-8602, Japan.

Contents

1. Introduction

Recently our understanding of chiral symmetry and gauge-field topology on the lattice has developed considerably. As an important development, the so-called overlap formalism was elaborated [1]. Moreover, the Ginsparg-Wilson relation [2] was rediscovered [3]. This led in particular to a gauge-invariant construction of abelian lattice chiral gauge theories [4] (see ref. [5] for a review on this subject). In this paper, we show that the Ginsparg-Wilson relation plays an important role also in odd dimensions.

A single massless Dirac fermion in odd dimensions is parity invariant on the classical level. However, in order to regularize it in a gauge invariant way, one has to break the parity invariance. For example, in the case of a Wilson fermion, the Wilson term breaks parity. One can obtain a massless fermion in the continuum limit by fine-tuning the hopping parameter, but the breaking of parity remains in most cases, which gives rise to the so-called parity anomaly [6].

The Ginsparg-Wilson relation, if formulated in a suitable form, can be used as a condition for lattice Dirac operators of massless fermions in odd dimensions just as in even dimensions. The fermion action with such a Dirac operator is manifestly invariant under a generalized parity transformation, which reduces to the ordinary parity transformation in the naive continuum limit. The fermion measure, however, transforms non-trivially under the generalized parity transformation. Thus the parity anomaly arises solely from the fermion measure in this formulation.

The connection to the even dimensional case can be clarified by first considering a Ginsparg-Wilson fermion in $2n$ dimensions and then performing a dimensional reduction down to $2n-1$ dimensions. The resulting system describes a *doublet* of massless Dirac fermions. By projecting out one of these fermions, we obtain a single odd-dimensional Ginsparg-Wilson fermion. The projection can be done in two different ways, leading to equivalent systems. The first type of projection can also be used for the Wilson fermion, but it has no counterpart in even dimensions. The second type of projection is possible only for the Ginsparg-Wilson fermion and it is analogous to the one used in constructing lattice chiral gauge theories in even dimensions. We study this analogy in detail and discuss in particular the relation to the overlap formalism. We also propose a natural definition of a lattice Chern-Simons term, which is consistent with odd dimensional Ginsparg-Wilson fermions.

For concreteness, we deal with three dimensions, which is relevant for most applications, in particular for high temperature superconductivity. However, the generalization to arbitrary odd dimensions is straightforward.

The organization of this paper is as follows. In section 2 we introduce the Ginsparg-Wilson relation in odd dimensions. We show that any Dirac operator satisfying the Ginsparg-Wilson relation is invariant under the generalized parity transformation. In section 3 we describe the dimensional reduction from D = 4 to D = 3 and the standard projection to a single Dirac fermion in the case of Wilson fermions. In section 4 we apply the dimensional reduction to Ginsparg-Wilson fermions. In section 5 we discuss the generalized parity invariance in the context of dimensional reduction and clarify its relation to the generalized chiral symmetry in even dimensions [7]. In section 6 we consider the analogy to the lattice construction of chiral gauge theories in even dimensions. In section 7 we comment on the relation to some previous results obtained from the overlap formalism. In section 8 we propose a lattice Chern-Simons term, which follows naturally from the previous considerations, and section 9 is devoted to our conclusions.

2. The Ginsparg-Wilson fermion in odd dimensions

The Ginsparg-Wilson relation in four dimensions is commonly written as[1]

$$D^{(4)}\gamma_5 + \gamma_5 D^{(4)} = a\, D^{(4)}\gamma_5 D^{(4)}\,, \tag{2.1}$$

where a is the lattice spacing. It turned out that this condition for the 4D lattice Dirac operator $D^{(4)}$ enables the physical properties related to chirality to be repre-

[1]The right-hand sides of eqs. (2.1), (2.3), (2.4) are understood as convolutions in c-space.

sented correctly on the lattice. It is crucial for $D^{(4)}$ to be local (with exponentially decaying correlations) and free of massless species doublers. Assuming as usual "γ_5-Hermiticity" for $D^{(4)}$,

$$D^{(4)\dagger} = \gamma_5 D^{(4)} \gamma_5 \,, \tag{2.2}$$

we can re-write the Ginsparg-Wilson relation in a form, which does not involve γ_5, and which is therefore compatible with odd dimensions,

$$D^{(4)} + D^{(4)\dagger} = a\, D^{(4)\dagger} D^{(4)} \,. \tag{2.3}$$

Hence we denote a 3D lattice Dirac operator $D^{(3)}$ as a Ginsparg-Wilson operator if it obeys

$$D^{(3)} + D^{(3)\dagger} = a\, D^{(3)\dagger} D^{(3)} \,, \tag{2.4}$$

and we expect it to describe a massless fermion in D = 3. As in even dimensions, a solution of this relation has to take the form

$$D^{(3)} = \frac{1}{a}\,(1 - V)\,, \tag{2.5}$$

where the operator V is *unitary*. In the naive continuum limit, V should turn into the identity operator. Explicit solutions for the 3D Ginsparg-Wilson operator are discussed in section 7.

The corresponding action

$$S = a^3 \sum_x \bar\psi(x) D^{(3)}(U) \psi(x)\,, \tag{2.6}$$

is invariant under

$$\psi(x) \mapsto i\,\mathcal{R}\,V\,\psi(x)\,; \qquad \bar\psi(x) \mapsto i\,\bar\psi(x)\,\mathcal{R}\,; \qquad U_\mu(x) \mapsto U^P_\mu(x)\,, \tag{2.7}$$

$$\text{or} \quad \psi(x) \mapsto i\,\mathcal{R}\,\psi(x)\,; \qquad \bar\psi(x) \mapsto i\,\bar\psi(x)\,V\,\mathcal{R}\,; \qquad U_\mu(x) \mapsto U^P_\mu(x)\,, \tag{2.8}$$

where $\mathcal{R}$ is the space-time reflection operator, $\mathcal{R}: \; x \to -x$. U^P is the parity transformed gauge configuration,

$$U^P_\mu(x) = U_\mu(-x - a\hat\mu)^\dagger\,, \tag{2.9}$$

$\hat\mu$ being a unit vector in μ direction. We assume that the 3D Ginsparg-Wilson operator $D^{(3)}$ has the property

$$D^{(3)}(U)^\dagger = \mathcal{R} D^{(3)}(U^P) \mathcal{R}\,, \tag{2.10}$$

which holds also for the 3D Wilson operator. If the right-hand side of the relation (2.4) would vanish — as is the case with the continuum massless Dirac operator $\sigma_\mu(\partial_\mu + iA_\mu(x))$ — then the action would be invariant under the ordinary parity transformation, which is obtained from eq. (2.7) or (2.8) by replacing V by the identity operator. The right-hand side of (2.4) being non-zero, the action (2.6) is not invariant under the ordinary parity transformation, but it still has exact invariance on the lattice under the generalized parity transformation (2.7) or (2.8), which reduces to the ordinary one in the naive continuum limit.

On the other hand, the fermion measure does change non-trivially under transformation (2.7) or (2.8),

$$\mathrm{d}\psi\, \mathrm{d}\bar{\psi} \mapsto (\det V)^{-1}\, \mathrm{d}\psi\, \mathrm{d}\bar{\psi}\,, \tag{2.11}$$

which gives rise to the parity anomaly in this formulation. This is in contrast to the situation with the Wilson fermion, where the parity breaking appears explicitly in the action through the Wilson term. A massless Wilson fermion is achieved only by fine-tuning the hopping parameter. On the other hand, for Ginsparg-Wilson fermions the pole of the propagator at zero momentum is preserved to all orders in perturbation theory, hence one obtains a single massless Dirac fermion without fine-tuning.

The situation is analogous to the generalized chiral symmetry in even dimensions [7]. Note that similar to the Nielsen-Ninomiya theorem, a local, undoubled lattice fermion in odd dimensions cannot obey the standard parity symmetry. This rule is circumvented by the lattice modified parity invariance (2.7) resp. (2.8). In order to clarify the relation to the even dimensional Ginsparg-Wilson fermions, we formally construct $D^{(3)}$ from $D^{(4)}$ by means of dimensional reduction and a subsequent projection. In the next section, we consider these operations in the case of Wilson fermions to establish the conceptual basis.

In passing, we comment that if $D^{(3)}$ satisfies the Ginsparg-Wilson relation, so does $D^{(3)\dagger}$. Therefore the action

$$S' = -a^3 \sum_x \bar{\psi}'(x) D^{(3)}(U)^\dagger \psi'(x) \tag{2.12}$$

is also expected to describe a massless Dirac fermion. However, the parity anomaly is represented by $\det(V^\dagger)$ instead of $\det V$. This manifests the well-known regularization dependence of parity anomaly. We will discuss this issue in more detail in section 7. We also note that the action (2.12) can be transformed to the form (2.6) by the field re-definitions $\psi'(x) = V\psi(x)$ and $\bar{\psi}'(x) = \bar{\psi}(x)$. However, the measure is different, $\mathrm{d}\psi'\, \mathrm{d}\bar{\psi}' = (\det V)^{-1}\, \mathrm{d}\psi\, \mathrm{d}\bar{\psi}$. Therefore, the two systems (2.6) and (2.12) with the standard measure are *not* equivalent.[2] In fact the two Dirac fermions form a parity doublet, as we will see in section 4.

[2]Note that in D = 4 the corresponding two systems are equivalent due to the γ_5-hermiticity (2.2). $D^{(3)}$, however, has the property (2.10) instead, which makes a crucial difference.

3. Dimensional reduction of Wilson fermions

We start from a 4D Wilson fermion with a lattice gauge field $U_\mu(x)$ on a 4D periodic lattice $L_1 \times \cdots \times L_4$. Then we perform a "dimensional reduction" in the framework of the lattice formulation to obtain a 3D system. By "dimensional reduction", we mean that we let $L_4 = 1$ and take the gauge field to be $U_4(x) = 1$ for any lattice site x. $L_1 \ldots L_3$ may be finite or infinite. The system we obtain is described by the action

$$S = a^3 \sum_x \bar{\Psi}(x)[D_{\rm w}(U) + m]\Psi(x)\,, \tag{3.1}$$

where $D_{\rm w}$ is given by

$$D_{\rm w} = \frac{1}{2}\sum_{\mu=1}^{3}\{\gamma_\mu(\nabla_\mu^* + \nabla_\mu) + a\nabla_\mu^*\nabla_\mu\}\,, \tag{3.2}$$

and the forward resp. backward lattice derivatives act as

$$\begin{aligned}
\nabla_\mu\Psi(x) &= \frac{1}{a}\Big[U(x,\mu)\Psi(x+a\mu) - \Psi(x)\Big] \\
\nabla_\mu^*\Psi(x) &= \frac{1}{a}\Big[\Psi(x) - U(x-a\hat{\mu})^{-1}\Psi(x-a\hat{\mu})\Big]\,.
\end{aligned}$$

For the 4D gamma matrices we use the representation

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix}; \qquad \gamma_4 = \begin{pmatrix} 0 & i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix}; \qquad \gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{3.3}$$

Note that $D_{\rm w}$ inherits γ_5-Hermiticity from the 4D Wilson operator,

$$D_{\rm w}(U)^\dagger = \gamma_5 D_{\rm w}(U)\gamma_5\,. \tag{3.4}$$

In addition, $D_{\rm w}$ has some properties which the 4D Wilson operator does not have. Firstly,

$$D_{\rm w}(U)\,\Gamma = \Gamma\, D_{\rm w}(U)\,, \tag{3.5}$$

[8] where Γ is a unitary and hermitean matrix defined as

$$\Gamma = i\gamma_4\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.6}$$

Due to commutation relation (3.5), the action (3.1) is invariant under

$$\Psi(x) \mapsto {\rm e}^{i\alpha\Gamma}\Psi(x)\,; \qquad \bar{\Psi}(x) \mapsto \bar{\Psi}(x){\rm e}^{-i\alpha\Gamma}\,; \qquad U_\mu(x) \mapsto U_\mu(x)\,, \tag{3.7}$$

where α is a real parameter. Under space-time reflection, $D_{\rm w}$ behaves as eq. (2.10),

$$D_{\rm w}(U)^\dagger = \mathcal{R}D_{\rm w}(U^P)\mathcal{R}\,. \tag{3.8}$$

Using the relations (3.4) and (3.8), we see that the action (3.1) is invariant under the (standard) parity transformation[3]

$$
\begin{aligned}
\Psi(x) &\mapsto \gamma_5 \Psi(-x) = \gamma_5 \mathcal{R}\Psi(x) \\
\bar{\Psi}(x) &\mapsto \bar{\Psi}(-x)\gamma_5 = \bar{\Psi}(x)\mathcal{R}\gamma_5 \\
U_\mu(x) &\mapsto U_\mu^P(x)\,.
\end{aligned} \tag{3.9}
$$

Due to property (3.5), the action (3.1) naturally decouples into two Dirac fermions, where $1/2(1 \pm \Gamma)$ acts as a projection operator. Let us introduce the unitary matrix $\bar{U}$ that diagonalizes Γ,

$$
\bar{U}^\dagger\,\Gamma\,\bar{U} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \gamma_5\,; \qquad \bar{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}. \tag{3.10}
$$

Using this unitary matrix $\bar{U}$, also the Wilson operator can be brought into a block-diagonal form as

$$
\bar{U}^\dagger D_{\mathrm{w}} \bar{U} = \begin{pmatrix} D_{\mathrm{w}}^{(3)} & 0 \\ 0 & D_{\mathrm{w}}^{(3)\dagger} \end{pmatrix}, \tag{3.11}
$$

which follows only from the properties (3.4) and (3.5). From the explicit form (3.2), we see that $D_{\mathrm{w}}^{(3)}$ in eq. (3.11) is given by the 3D Wilson operator

$$
D_{\mathrm{w}}^{(3)} = \frac{1}{2}\sum_{\mu=1}^{3}\{\sigma_\mu(\nabla_\mu^* + \nabla_\mu) + a\nabla_\mu^*\nabla_\mu\}\,. \tag{3.12}
$$

By decomposing the fermion fields as

$$
\Psi(x) = \bar{U}\begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}, \qquad \bar{\Psi}(x) = (\bar{\psi}(x), \bar{\psi}'(x))\,\bar{U}^\dagger\,, \tag{3.13}
$$

we find that the action (3.1) decouples into two 3D Dirac fermions, with mass and Wilson parameter of opposite signs. Note also that the integration measure decouples simply as

$$
\mathrm{d}\Psi\,\mathrm{d}\bar{\Psi} = \mathrm{d}\psi\,\mathrm{d}\bar{\psi}\cdot\mathrm{d}\psi'\,\mathrm{d}\bar{\psi}'\,, \tag{3.14}
$$

without a nontrivial jacobian factor. The unitary transformation performed here corresponds to a change of the representations of the 4D γ matrices.

One can extract one of the two 3D Dirac fermions by imposing the constraint

$$
\Gamma\Psi = \Psi\,; \qquad \bar{\Psi}\Gamma = \bar{\Psi}\,. \tag{3.15}
$$

However, this constraint is not compatible with the parity transformation (3.9), since Γ does not commute with γ_5. Therefore, parity invariance (3.9) does *not* survive the

[3]In transformation (3.9), γ_5 could be replaced by γ_4. More generally, one can replace γ_5 by $\gamma_5 e^{i\alpha\Gamma}$, as we see if we combine the transformations (3.7) and (3.9). However, we stay with (3.9), because the further generalizations do not provide additional insight.

projection (3.15). This can be understood intuitively since the parity transformation (3.9) exchanges the two 3D Dirac fermions, hence it cannot be described by using only one of them.

We also remark that the existence of the (standard) parity invariance (3.9) in the 3D *doublet* system does not imply the masslessness of the two Dirac fermions. It only implies that they have masses of opposite signs with the same magnitude. It is for a *single* 3D Dirac fermion that the parity invariance on the classical level implies masslessness.

4. 3D Ginsparg-Wilson fermions from dimensional reduction

Let us now consider a 4D Ginsparg-Wilson fermion and perform a dimensional reduction to $\mathrm{D} = 3$. The system we obtain is described by the action

$$S = a^3 \sum_x \bar{\Psi}(x) D(U) \Psi(x)\,, \tag{4.1}$$

where D satisfies the Ginsparg-Wilson relation

$$D + D^\dagger = a\, D^\dagger D\,. \tag{4.2}$$

We also assume that it fulfills the properties (3.4), (3.5) and (3.8).

Since D has all the properties of D_{w} that we used in section 3, all the statements in the previous Section apply to the present case. Hence the system (4.1) is invariant under the (standard) parity transformation (3.9) and it decouples into two Dirac fermions if we apply $\frac{1}{2}(1 \pm \Gamma)$ as a projector. We can bring the operator D into a block-diagonal form as

$$\bar{U}^\dagger\, D\, \bar{U} = \begin{pmatrix} D^{(3)} & 0 \\ 0 & D^{(3)\dagger} \end{pmatrix}. \tag{4.3}$$

Due to the 3D Ginsparg-Wilson relation for the doublet, eq. (4.2), the operator $D^{(3)}$ in eq. (4.3) obeys the 3D singlet Ginsparg-Wilson relation (2.4). By using the decomposition (3.13), we obtain the systems (2.6) and (2.12) with the standard integration measure. Therefore, the system (4.1) describes a doublet of 3D Ginsparg-Wilson fermions, which are exchanged under the standard parity transformation (3.9). One can extract one of the two 3D Ginsparg-Wilson fermions by imposing the constraint (3.15).

In analogy to the even dimensional case [4], we define the operator $\hat{\gamma}_5$,

$$\hat{\gamma}_5 = \gamma_5(1 - aD)\,, \tag{4.4}$$

which is hermitean,

$$\hat{\gamma}_5^\dagger = \hat{\gamma}_5\,, \tag{4.5}$$

due to the γ_5-hermiticity of D. The Ginsparg-Wilson relation (4.2) further implies

$$(\hat{\gamma}_5)^2 = 1\,,$$
$$\gamma_5 D = -D\hat{\gamma}_5\,. \tag{4.6}$$

Eqs. (4.5) and (4.6) show that $\hat{\gamma}_5$ is also unitary. In addition, eq. (4.6) implies

$$\mathrm{e}^{i\alpha\gamma_5} D \mathrm{e}^{i\alpha\hat{\gamma}_5} = D\,, \qquad \mathrm{e}^{i\beta\hat{\gamma}_5} D^\dagger \mathrm{e}^{i\beta\gamma_5} = D^\dagger\,, \tag{4.7}$$

which is analogous to the generalized lattice chiral symmetry in even dimensions [4].

In the present 3D case, the operator $\hat{\gamma}_5$ has the additional property

$$\hat{\gamma}_5\,\Gamma = -\Gamma\,\hat{\gamma}_5\,, \tag{4.8}$$

which follows from relation (3.5). Eqs. (4.5), (4.6) and (4.8) imply that $\hat{\gamma}_5$ has the structure

$$\hat{\gamma}_5 = \begin{pmatrix} X & iY \\ -iY & -X \end{pmatrix}, \tag{4.9}$$

where X, Y are hermitean matrices satisfying $X^2 + Y^2 = 1$ and $XY = YX$. Solving (4.4) for D, one obtains

$$D = \frac{1}{a}\begin{pmatrix} 1-X & -iY \\ -iY & 1-X \end{pmatrix}, \tag{4.10}$$

and eq. (4.3) implies

$$D^{(3)} = \frac{1}{a}\left\{1 - (X + iY)\right\}. \tag{4.11}$$

Thus the 3D Ginsparg-Wilson operator $D^{(3)}$ takes the general form (2.5) with the unitary operator V given by $V = X + iY$.

5. Generalized parity transformations

As in the case of the Wilson fermion, the projection (3.15) is not compatible with the parity transformation (3.9). However, we will see that the action (4.1) is actually invariant under a more general class of parity transformations due to the Ginsparg-Wilson relation. Then we are going to show that there exist generalized parity transformations, which *are* compatible with the projection (3.15). In the next section, we will see that again due to the Ginsparg-Wilson relation, there exists a different type of projection, which is compatible with the standard parity transformation (3.9).

Let us start from the ansatz

$$\Psi(x) \mapsto \mathcal{R}A\Psi(x)$$
$$\bar{\Psi}(x) \mapsto \bar{\Psi}(x)B\mathcal{R}$$
$$U_\mu(x) \mapsto U_\mu^P(x)\,, \tag{5.1}$$

for a generalized parity transformation, where A and B are operators acting on the fermion field. Requiring the action (4.1) to be invariant under transformation (5.1), and using the property (3.8), we obtain the condition

$$BD^{\dagger}A = D\,. \tag{5.2}$$

The ordinary parity transformation (3.9) corresponds to the particular case $A = B = \gamma_5$, which indeed satisfies the condition (5.2), cf. relation (3.4).

Exploiting the properties (4.7), we obtain a more general solution to the condition (5.2) as

$$A = \mathrm{e}^{i\beta\gamma_5}\gamma_5\mathrm{e}^{i\alpha\hat{\gamma}_5}\,, \qquad B = \mathrm{e}^{i\alpha\gamma_5}\gamma_5\mathrm{e}^{i\beta\hat{\gamma}_5}\,, \tag{5.3}$$

with real parameters α, β. The invariance of the projection (3.15) under the transformation (5.1) requires that the operators A and B commute with Γ. Such a set of operators A, B can be obtained by choosing $\alpha = \pi/2$, $\beta = 0$ or $\alpha = 0$, $\beta = \pi/2$ in the generalized solution (5.3). These two choices correspond to

$$\text{(i)} \qquad A = i\gamma_5\hat{\gamma}_5\,, \quad B = i\,, \tag{5.4}$$

$$\text{(ii)} \qquad A = i\,, \quad B = i\gamma_5\hat{\gamma}_5\,, \tag{5.5}$$

respectively. Correspondingly, the projected fermion fields $\psi(x)$ and $\bar{\psi}(x)$ transform as (2.7) resp. (2.8) under the generalized parity transformation (5.1).

Hence the existence of the generalized parity transformation (2.7) or (2.8) in the action (2.6) can be understood in analogy to the generalized chiral symmetry in even dimensions.

6. Analogy to chiral gauge theories in D = 4

Due to property (4.6), the system (4.1) also allows for another projection by imposing the constraint

$$\hat{\gamma}_5\Psi = \Psi\,; \qquad \bar{\Psi}\gamma_5 = -\bar{\Psi}\,, \tag{6.1}$$

instead of the constraint (3.15). This is in fact the projection used to construct lattice chiral gauge theories in the even dimensions [4].

The projection (6.1) is actually compatible with the standard parity transformation (3.9), since

$$\hat{\gamma}_5(U^P)\,(\mathcal{R}\gamma_5) = (\mathcal{R}\gamma_5)\,\hat{\gamma}_5(U)\,. \tag{6.2}$$

The parity transformation written in terms of the projected fields $\psi(x)$ and $\bar{\psi}(x)$ is given by eq. (2.7).

Next we note that eq. (4.8) implies that $\hat{\gamma}_5$ has an equal number of eigenvalues $+1$ and -1. Consequently, the counterpart of the index in D = 4 vanishes identically,

$$\mathrm{Tr}\left(\frac{1+\gamma_5}{2}\right) - \mathrm{Tr}\left(\frac{1-\hat{\gamma}_5}{2}\right) = \frac{1}{2}\mathrm{Tr}\,(\gamma_5 D) = 0\,, \tag{6.3}$$

where the trace Tr is taken over all the spinor, space-time and gauge indices. Hence it is possible to diagonalize $\hat{\gamma}_5$, for instance as

$$\tilde{U}^\dagger \hat{\gamma}_5 \tilde{U} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \gamma_5 \,, \qquad \tilde{U} = \bar{U} \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} \bar{U}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -V^\dagger \end{pmatrix} . \tag{6.4}$$

Using this matrix, we can bring the operator D into a block-diagonal form,

$$\Gamma D \tilde{U} = \begin{pmatrix} D^{(3)} & 0 \\ 0 & D^{(3)\dagger} \end{pmatrix} . \tag{6.5}$$

By decomposing the fermion fields as

$$\Psi(x) = \tilde{U} \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix} , \qquad \bar{\Psi}(x) = (\bar{\psi}(x), \bar{\psi}'(x)) \, \Gamma \,, \tag{6.6}$$

we see that the action (4.1) decouples into two 3D Ginsparg-Wilson fermions. Note also that the integration measure decouples as in eq. (3.14), without a nontrivial jacobian factor, since $\det \Gamma \det \tilde{U} = 1$. Although the decomposition (6.6) is different from (3.13), the resulting systems are equivalent.

In ref. [9], the 4D Weyl fermion belonging to the real representation of the gauge group has been studied using the Ginsparg-Wilson fermion. There, it was shown that the fermion measure can be defined in a gauge invariant way, although the construction is quite different from the one presented here.

7. Relation to the overlap formalism

So far, we have not specified a particular form of the Ginsparg-Wilson operators. In even dimensions, an explicit example of the Ginsparg-Wilson operator $D^{(4)}$, which satisfies the Ginsparg-Wilson relation (2.1) and other required properties, is given by the overlap operator [10]. Up to moderate coupling strength, the overlap operator is local and free of doubling and mass renormalization[4]. The operator D, obtained by dimensional reduction of an overlap operator $D^{(4)}$, is given by

$$D = \frac{1}{a} \left\{ 1 - A_{\rm w} / \sqrt{A_{\rm w}^\dagger A_{\rm w}} \right\} , \qquad A_{\rm w} = 1 - a D_{\rm w} \,, \tag{7.1}$$

where the Wilson operator $D_{\rm w}$ is defined in eq. (3.2). As in the even dimensional case [5, 13], the construction presented in section 6 becomes mathematically isomorphic to the overlap formalism [1].

[4]The collapse of the latter two properties was observed in a strong coupling expansion [11]. For instance, in quenched QCD_4 the applicability of the overlap formula — starting from $D_{\rm w}^{(4)}$ — extends down to about $\beta \approx 5.6$ [12].

The overlap formalism, which was originally proposed to construct a lattice chiral gauge theory,[5] has also been applied to odd dimensions [15, 16]. In ref. [15] it was pointed out that the overlap formalism provides a lattice regularization of a single 3D Dirac fermion with manifest parity invariance except for the phase of the fermion determinant. This is due to the fact that the auxiliary many-body hamiltonian, which plays the central role in the overlap formalism, has a manifest parity invariance. Explicitly, the many-body hamiltonian is given by

$$\mathcal{H}(U) = \sum_x (\xi(x)^\dagger, \eta(x)^\dagger)\, H(U) \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}, \tag{7.2}$$

with the hermitean operator

$$H = \gamma_5 \,(1 - a\, D_{\rm w})\,. \tag{7.3}$$

The many-body hamiltonian (7.2) has a manifest parity invariance

$$\xi(x) \mapsto i\,\xi(-x)\,; \qquad \eta(x) \mapsto i\,\eta(-x)\,; \qquad U_\mu(x) \mapsto U^P_\mu(x)\,. \tag{7.4}$$

In fact one can re-write the overlap formalism in a path integral form following e.g. ref. [17] to arrive at the formalism in section 6. Then one observes that the parity transformation (7.4) maps to the standard parity transformation (3.9), which survives the projection (6.1).

For the particular choice (7.1), the corresponding 3D Ginsparg-Wilson operator $D^{(3)}$ defined through (4.3) or (6.5) takes the form

$$D^{(3)} = \frac{1}{a}\left\{1 - A^{(3)}_{\rm w} \Big/ \sqrt{A^{(3)\dagger}_{\rm w} A^{(3)}_{\rm w}}\right\} \tag{7.5}$$

$$A^{(3)}_{\rm w} = 1 - aD^{(3)}_{\rm w}\,. \tag{7.6}$$

This coincides with the overlap Dirac operator first dervived by Kikukawa and Neuberger [16]. The phase of the effective action $\det(D^{(3)})$ is parity odd, and it is half of the phase of $\det(A^{(3)}_{\rm w})$. The latter has been calculated in the continuum limit [18], and it turned out that the correct parity anomaly — described by the Chern-Simons term — is reproduced.

In general, infinitely heavy Dirac fermions in odd dimensions do not decouple completely. They leave a remnant in the effective action,[6] which can by expressed again by a Chern-Simons term. When one regularizes a single massless Dirac fermion, one has to introduce a number of heavy fermions with mass of the cutoff scale, such as Pauli-Villars regulators or massive doublers for the Wilson fermion. The parity

[5] See, however, ref. [14], where the problem of the so-called Wigner-Brillouin phase choice proposed in ref. [1] has been clarified.

[6] This is not too surprising, since the remnant quantity appears in terms of a local operator.

anomaly represented by the induced Chern-Simons term therefore depends on the number of heavy fermions introduced in the regularized theory. The coefficient of the Chern-Simons term has an ambiguity labelled by an integer, which specifies a universality class. In particular, ref. [18] considers the effective gauge action after integrating out the fermion fields, $\Gamma[A]$. In the limit of infinite fermion mass it amounts to

$$\lim_{m\to\infty} \Gamma[A] = 2n\pi i e^2 I[A]\,, \tag{7.7}$$

where e is the charge, $I[A]$ is the Chern-Simons term, and $n \in \mathbb{Z}$. It is related to the massless limit as

$$\lim_{m\to 0} \operatorname{Im} \Gamma[A] = (2n+1)\pi e^2 I[A] + \pi h[A]\,, \tag{7.8}$$

where $h[A]$ is an integer, which vanishes for smooth gauge fields. The integer n labels the universality classes. For the Wilson fermion, they were first studied in ref. [19]. The above choice (7.5) with (7.6) reproduces one of these classes.

A natural question is whether we can also reproduce other classes by using Ginsparg-Wilson fermions. At this point, we remark that the overlap solution to the Ginsparg-Wilson relation can be generalized in two respects [20]: first, the relation (2.4) itself can be generalized to

$$D^{(3)} + D^{(3)\dagger} = 2aD^{(3)\dagger} R D^{(3)}\,, \tag{7.9}$$

where the kernel R has to be local and it must not be parity-odd.[7] This relation is solved by the generalized overlap formula

$$D^{(3)} = \frac{1}{2a}\frac{1}{\sqrt{R}}\left(1 - A^{(3)}/\sqrt{A^{(3)\dagger}A^{(3)}}\right)\frac{1}{\sqrt{R}}\,, \qquad A^{(3)} = 1 - 2a\sqrt{R}D_{\rm w}^{(3)}\sqrt{R}\,. \tag{7.10}$$

In addition, we also have much freedom in the choice of $A^{(3)}$: the Wilson operator $D_{\rm w}^{(3)}$ in its definition can be replaced by many other lattice Dirac operators $D_0^{(3)}$. If $D_0^{(3)}$ is local and free of massless doublers, we arrive at a sensible Ginsparg-Wilson operator $D^{(3)}$. (However, $D_0^{(3)}$ does not need to be parity invariant in any sense.)

If we now vary either R, or $D_0^{(3)}$ or both, we can change the universality class. In fact, *all* classes are accessible in this way, so the set of 3D Ginsparg-Wilson fermions covers all of them. To demonstrate this, we consider as an example the form

$$A^{(3)} = (1 - aD_{\rm w}^{(3)})\,(1 - 2a\,D_{\rm w}^{(3)})^{2n}\,, \qquad R_{x,y} = \frac{1}{8n+2}\,\delta_{x,y}\,, \tag{7.11}$$

where $n = 0, 1, 2, \dots$ The phase of $\det(A^{(3)})$ for these cases has been calculated in the continuum limit [18], and the resulting parity anomaly is $(2n+1)$ times the one

[7] As in even dimensions, also perfect [2] and classically perfect [21] actions are solutions to the Ginsparg-Wilson relation. In their construction, the term R^{-1} appears in the renormalization group transformation term, and the standard choice $R_{x,y} = \frac{1}{2}\delta_{x,y}$ is optimal for locality of the free fermion [22].

obtained for the standard choice (7.6). Moreover, we can switch from $D^{(3)}$ to $-D^{(3)\dagger}$ (cf. Section 2), which is a Ginsparg-Wilson operator for a kernel R with opposite sign. This changes also the sign of the parity anomaly. Therefore, by exploiting the freedom in choosing R and D_0, we obtain Ginsparg-Wilson fermions in all the universality classes — labelled by an integer ranging from $-\infty$ to $+\infty$ — which were identified before for Wilson fermions [18].

This confirms that we are not dealing with an anomaly in the usual sense, where the amount is the same for all correct regularizations. In the case of the parity anomaly, an infinite number of universality classes coexist on the same level.

8. A lattice formulation of the Chern-Simons term

Finally, the parity anomaly obtained from the 3D Ginsparg-Wilson fermion suggests that

$$e^{iS_{\rm CS}} \stackrel{\rm def}{=} \frac{\det(A_{\rm w}^{(3)})}{|\det(A_{\rm w}^{(3)})|}, \tag{8.1}$$

where we use $A_{\rm w}^{(3)}$ as defined in eq. (7.6), can be used as a natural definition of the Chern-Simons term on the lattice. Of course, a more general operator $A^{(3)}$ can be used, as long as we stay in the same universality class as the one obtained for $A_{\rm w}^{(3)}$ in a given gauge configuration.

The normalization of $S_{\rm CS}$ is chosen in such a way that it transforms as

$$S_{\rm CS} \mapsto S_{\rm CS} + 2\pi\nu\,, \tag{8.2}$$

under a gauge transformation, where ν is the winding number characterizing the gauge transformation. Consequently, the phase factor $e^{iS_{\rm CS}}$ should be gauge invariant, and indeed the right-hand side of eq. (8.1) is manifestly gauge invariant. Since the index theorem holds in even dimensions on the lattice for Ginsparg-Wilson fermions [21], one may expect that the above definition captures the topological aspects of gauge theories.[8] It is an interesting question to study this term in the light of the No Go theorem established in ref. [23]. Finally, we note that a related Chern-Simons lattice term has been suggested in ref. [24].

9. Discussion and outlook

In this paper, we have introduced the Ginsparg-Wilson relation in odd dimensions and we revealed its importance. We have shown that an action based on any lattice

[8]The term in eq. (8.1) is not defined if the spectrum of $A^{(3)}$ includes a zero eigenvalue. Note that in this case, the Dirac operator $D^{(3)}$ (given in eq. (7.5)) is not defined either. However, the set of configurations where this happens (the boundaries of topological sectors) is statistically negligible [10].

Dirac operator satisfying the Ginsparg-Wilson relation is invariant under a generalized parity transformation. The fermion measure, however, breaks parity invariance, giving rise to the parity anomaly. This is analogous to the chiral symmetry in even dimensions: there the chiral anomaly arises from the measure in the case of Ginsparg-Wilson fermions [7]. Thus, we naturally realize Fujikawa's approach to anomalies [25] at a fully regularized level on the lattice. This is in contrast to the situation with the Wilson fermion, where the parity breaking appears explicitly in the action through the Wilson term.

We note that the way the parity anomaly appears is similar to the situation discussed in ref. [26], where a single Dirac fermion is considered in the continuum with Pauli-Villars regularization. After integrating out the Pauli-Villars regulator, a (non-local) field re-definition of the fermion field is performed to make the action parity invariant. The parity anomaly arises from the jacobian of the field re-definition.

Let us finally give some possible directions for future studies. We can impose a Majorana condition on the 3D Ginsparg-Wilson fermion to obtain a massless 3D Majorana fermion, which can be used to construct supersymmetric $\mathcal{N} = 1$ Yang-Mills theory in three dimensions on the lattice. This provides a considerable simplification to the idea presented in ref. [27] using the overlap formalism. In ref. [28], it was suggested that parity anomaly in odd dimensions may be related to a CPT anomaly in chiral gauge theories in even dimensions. Indeed, according to ref. [29] a CPT anomaly exists in the exact solution [14] of two-dimensional $\mathrm{U}(1)$ chiral gauge theories. We hope that our results may also shed light on that issue. We have further proposed a possible definition of the Chern-Simons term on the lattice. It turns into the correct form in the continuum limit. As the index theorem provides a sensible definition of the topological charge on the lattice for Ginsparg-Wilson fermions, it is conceivable that the proposed definition of the Chern-Simons term captures the topological aspect of gauge theory.

To conclude, we note that the consistency of the Ginsparg-Wilson fermion revealed in this paper provides further support for the new approach to lattice chiral gauge theories and extends the range of its application to odd dimensions.

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